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ディィリクレ問題の安定領域

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Stable Domains in the Dirichlet Problem

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1. Introduction

The stability of the Dirichlet problem concerns the question to clarify how generalized solutions H_f^M of the Dirichlet problem on regions M with continuous boundary data f vary according to the small perturbation of boundaries ∂M of regions M (cf. e.g. [8], [9], [7], [1], etc.). The stability question is particularly important in various practical applications. For example, electric condensers play essential roles in the hard part of computer sciences which are applied to almost all electric instruments, which are used in many cases under the situations not free from the vibration. Perturbations of electrodes of condensers naturally effect more or less voltages of electrodes and then the stable functions of condensers are disturbed. The study of stability problem is thus of compelling importance not only in the theoretical aspect but also in the application view point. The purpose of this paper is to show that the Dirichlet problem is stable inside the region M if and only if the set of points in ∂M at which the complement $\overline{M}^c$ of the closure $\overline{M}$ of M is thin is of capacity zero. Originally we did not expect and even dreamt of the validity of the above result at all. Before we got the result we were studying the theory of time independent Schrödinger equations with measure potentials of Kato class. While checking a situation related to the Dirichlet problem of the above equation, we came to a conclusion, which by chance suggested the possibility of producing the above result. It may be of some interest to observe that we will use the theory of Schrödinger equations with measure potentials of Kato class which superficially appears to have nothing to do with the stability question to prove the above result. We lectured on this result at several places such as Research Institute of Mathematical Analysis at Kyoto University. Although we did not have published the above result in any professional journal except publicizing it by lectureing or by distributing the preprints, a French mathematician Professor Arnaud de la Pradelle wrote to us that he suspected that the proof of the result might be shortened if we use the blayage theory instead of the theory of Schrödinger equations. Thanks to his suggestion we obtained a direct and simpler alternative proof.

We denote by R^d the Euclidean space of dimension $d \geq 2$ and by $\overline{\mathrm{R}}^d = \mathrm{R}^d \cup \{\infty\}$ the one point compactification of R^d. A *Carathéodory domain* M in $\overline{\mathrm{R}}^d$ is a connected open subset of $\overline{\mathrm{R}}^d$ such that $\partial \overline{M} = \partial M$. We will only consider those Carathéodory domains M satisfying $\partial M \subset \mathrm{R}^d$, which is not an essential restriction. We say that a sequence of domains $(M_n)_{n>1}$ is a *squeezer* of $\overline{M}$ if M_n are subdomains of $\overline{\mathrm{R}}^d$ with $\partial M_n \subset \mathrm{R}^d$ such that

$$M_n \supset \overline{M}, \quad \partial M_n \to \partial M.$$

The latter means that for any $\delta > 0$ we can specify a number $n(\delta)$ such that for all $n \geq n(\delta)$ each of sets ∂M_n and ∂M lies in a δ-neighborhood of another. By $\partial \overline{M} = \partial M$ such a squeezer exists.

Any $f \in C(\partial M)$ can be extended to an $f \in C(\overline{\mathrm{R}}^d)$. Based upon the fact that any $f \in C(\overline{\mathrm{R}}^d)$ can be uniformly approximated on a neighborhood of $\overline{M}$ by a difference of two continuous superharmonic functions, we can easily show the existence of the *external Dirichlet solution*

$$H_f^{\overline{M}}(x) := \lim_{n \to \infty} H_f^{M_n}(x) \quad (x \in \overline{M})$$

independent of the choice of the squeezer $(M_n)_{n>1}$ of $\overline{M}$ and the extension to $\overline{\mathrm{R}}^d$ of $f \in C(\partial M)$. The convergence is almost uniform on M and hence $H_f^{\overline{M}}$ is harmonic on M. The Dirichlet problem is said to be *stable* in M if $H_f^{\overline{M}} = H_f^M$ on M for any $f \in C(\partial M)$. A point $y \in \partial M$ is referred to as a *stable point* if $H_f^{\overline{M}}(y) = f(y)$ for every $f \in C(\partial M)$. We denote by

$$\sigma M$$

the totality of stable points y in ∂M. Any point $y \in \partial M \setminus \sigma M$ is called naturally an *unstable point*. It is known (cf. [8], [9, p.308]) that $y \in \sigma M$ if and only if the complement $\overline{M}^c$ of $\overline{M}$ is thick (i.e. not thin) at y. The follwing is a famous old result (cf. [8], [9, p.340]):

The Keldysh theorem. *The Dirichlet problem is stable in M if and only if $\partial M \setminus \sigma M$ is of harmonic measure zero with respect to M.*

Here the harmonic measure $\omega(\cdot; \partial M \setminus \sigma M, M)$ of the set $\partial M \setminus \sigma M$, which will be seen to be a Borel set, relative to the domain M is the generalized solution $H^M_{\chi_{\partial M \setminus \sigma M}}$ of the Dirichlet problem on M with the boundary data $\chi_{\partial M \setminus \sigma M}$, the characteristic function of the set $\partial M \backslash \sigma M$. The *purpose* of this paper is to maintain that the condition for $\partial M \backslash \sigma M$ to have zero harmonic measure in the above Keldysh theorem can be replaced by the condition for $\partial M \backslash \sigma M$ to have zero capacity. Here the capacity means the logarithmic capacity for $d = 2$ and the Newtonean capacity for $d \geq 3$. Namely, we will prove the following result :

Theorem. *The Dirichlet problem is stable in* M *if and only if* $\partial M \backslash \sigma M$ *is of capacity zero*.

Hence we can conclude that the following three conditions are equivalent by pairs : (i) the Dirichlet problem is stable in M ; (ii) $\partial M \backslash \sigma M$ is of harmonic measure zero with respect to M ; (iii) $\partial M \backslash \sigma M$ is of capacity zero.

The domain M in the above theorem is either bounded in R^d or contains the point at infinity as its interior point. In the latter case the domain M can be transformed to a bounded domain by an inversion without changing the content of the above theorem by using the Kelvin transform. For this reason, hereafter, we may assume without loss of generality that M is a *bounded* Carathéodory domain in R^d, $d \geq 2$.

After finishing this §1 *Introduction* we will give two kinds of proofs of our main theorem mentioned above. First in §2 titled *The proof based on the theory of Schrödinger equations*, we will present our original proof of the theorem using certain results on Schrödinger equations of measure potentials of Kato class, from which in reality we discovered the above result. Secondly in §3 titled *The proof based on the theory of balayage*, we will prove the theorem by using the balayage method, or sweeping out method, which is suggested as stated above by Professor de la Pradelle to whom we are deeply grateful. We had several occasions to give lectures on our main theorem as mentioned above at Osaka City University, Kyoto University, Kyoto Sangyo University, and Ochanomizu Women's University. We record here in §4 titled *APPENDIX : An abstract for one hour talk* the exact text of the abstract distributed to the audience for one hour lecture at Department of Mathematics, Faculty of Science, Osaka City University written in Japanese on our original approach to discover and prove of our main theorem above.

2. The proof based on the theory of Schrödinger equations

No matter what kind of approach we may follow in the stability question it seems inevitable to introduce and examine the external Green function in addition to the usual or internal, so to speak, Green function. In subsection 2.1. *External Green fubctions*, we give the definition of external Green functions and state their properties which are necessary for the proof in this section. The importance of Schrödinger equations lies in the fact that their solutions form the Brelot space, i.e. the typical harmonic space associated with second order elliptic partial differential equations. In subsection 2.2. *Brelot spaces of Schrödinger equations*, an outline of the theory of the object in this title with measure potentials of Kato class is presented which we use in the proof of this section. In the final subsection 2.3. *Proof of Theorem*, we will give a proof of our theorem using Schrödinger equations of measure potentials of Kato class. This is probably the first example of efficient application of the above equations to certain other potential theoretic result thanks to the generalization of potentials of equations from those of absolutely continuous case to those of singular case.

2.1. External Green functions. Let g be the fundamental harmonic function on R^d, i.e. $g(0)=+\infty$ and, for $x\neq 0$, $g(x)=\log(1/|x|)(d=2)$ and $g(x)=1/|x|^{d-2}(d\geq 3)$. The logarithmic $(d=2)$ or Newtonean $(d\geq 3)$ potential U^μ of a Radon measure μ (in general signed) is given by

$$U^\mu(x):=g*\mu(x)=\int_{\mathrm{spt}\mu} g(x-y)\,d\mu(y)$$

as far as it is meaningful, where spt μ is the support of μ.

Take a bounded Carathéodory domain M. In addition to its usual Green function $G(x,y)=G^M(x,y)$ on M we can consider its *external Green function* $G^*(x,y)$. It can be mainly defined by either one of the two mutually equivalent methods. The first definition uses a squeezer $(M_n)_{n\geq 1}$ of $\overline{M}$ and the associated sequence of Green functions $G_n(x,y)$ of M_n. Then we define $G^*(x,y)$ by

(2.1) $$G^*(x,y):=\lim_{n\to\infty} G_n(x,y) \qquad ((x,y)\in \overline{M}\times\overline{M}).$$

The limit function does not depend on the choice of squeezers $(M_n)_{n\geq 1}$ and is determined uniquely, which can be seen instantly by the principle of enlarging domains for Green functions. To deduce various properties of $G^*(x,y)$ it is convenient to choose $(M_n)_{n\geq 1}$ as to satisfy $\overline{M}_{n+1}\subset M_n$ and M_n being regular. Then $G_n(x,y)\downarrow G^*(x,y)$ on $\overline{M}\times\overline{M}$. By this we see that $x\longmapsto G^*(x,y)$ is harmonic on $M\setminus\{y\}$ for each fixed $y\in M$; $G^*(x,y)>0$ on M for each fixed $y\in M$. As a result of the symmetry of $G_n(x,y)$ we also have the symmetry of $G^*(x,y)$:

$$G^*(x,y)=G^*(y,x) \qquad ((x,y)\in\overline{M}\times\overline{M}).$$

Since $G_n(x,y)\geq G^*(x,y)$ on $M\times M$, we see that

$$G^*(x,y)\geq G(x,y) \qquad ((x,y)\in M\times M).$$

Since $G_1(x,y) \geq G^*(x,y) \geq G(x,y)$ on $M \times M$, we see that $G^*(\cdot,y)$ has the fundamental pole at $y \in M : x \longmapsto G^*(x,y) - g(x-y)$ is harmonic at $y \in M$.

We state another definition of $G^*(x,y)$ different from but equivalent to (2.1). Let ε_y be the Dirac measure with support at $y \in \overline{M}$ and $\beta_{\overline{M}^c}$ be the balayage operator for measures on $\overline{M}$ to the complement $\overline{M}^C$ of $\overline{M}$. Then $G^*(x,y)$ can also be given by

$$G^*(x,y) := U^{\varepsilon_y}(x) - U^{\beta_{\overline{M}^c}\varepsilon_y}(x) \qquad ((x,y) \in \overline{M} \times \overline{M}). \tag{2.2}$$

By this representation and some detailed properties of the balayaged measure $\beta_{\overline{M}^c}\varepsilon_y$, we can see the follwoing important boundary behaviors of $G^*(\cdot,y)$ (cf. [9, p.333]) : for any fixed $b \in M$, $a \in \partial M$ satisfies

$$G^*(a,b) = 0 \tag{2.3}$$

if and only if $a \in \sigma M$; equivalently, $a \in \sigma M$ satisfies

$$G^*(a,b) > 0 \tag{2.4}$$

if and only if $a \in \partial M \backslash \sigma M$; concerning (2.3), the function $(x,y) \longmapsto G^*(x,y)$ is continuous at (a,b) as the function on $\overline{M} \times M$.

We insert here a remark that a part of the above fact can slightly be generalized as follows : (2.3) is valid not only for $a \in \sigma M$ and $b \in M$ but also for $a \in \sigma M$ and $b \in \overline{M} \backslash \{a\}$; the function $(x,y) \longmapsto G^*(x,y)$ is continuous at $(a,b) \in \sigma M \times (\overline{M} \backslash \{a\})$ as the function on $\overline{M} \times \overline{M}$. To prove this it is sufficient to treat the case $b \in \partial M \backslash \{a\}$. Let $B := B(b,r) = \{|x-b| < r\}$ and we chooser $r > 0$ so small that $a \notin \overline{B}$. Consider $M' = \overline{M \cup B} \backslash \partial(\overline{M \cup B})$. We see that M' is also a bounded Carathéodory domain such that $M' \supset M \supset (\overline{M} \cup B)$. We denote by $G'^*(x,y)$ the external Green function of M'. Observe that $a \in \sigma M$ is a local property (cf. e.g. the Wiener type criterion for $a \in \sigma M$ ([8], [9, p.287])). Hence we can conclude that $a \in \sigma M'$. Since $b \in \sigma M'$, (2.3) assures that $G'^*(a,b) = 0$, and the function $(x,y) \longmapsto G'^*(x,y)$ is continuous at (a,b) as the function on $\overline{M}' \times \overline{M}'$. From the fact that $0 \leq G^*(x,y) \leq G'^*(x,y)$ it follows that $G^*(a,b)$ and that the function $(x,y) \longmapsto G^*(x,y)$ is continuous at (a,b) as the function on $\overline{M} \times \overline{M}$.

2.3. Brelot spaces of Scrödinger equations. Consider a Radon measure μ (in general signed) on $\mathbb{R}^d$. Consider a stationary Schrödinger equation with its potential μ :

$$(-\Delta + \mu)u = 0. \tag{2.5}$$

Any *continuous* distributional solution u of (2.5) on an open set D of $\mathbb{R}^d$ will be referred to as a *μ-harmonic function* on D and the totality of μ-harmonic functions u on D will be denoted by ${}_{\mu}H(D)$. Then ${}_{\mu}H : D \longmapsto {}_{\mu}H(D)$ defines a sheaf (usually referred to as a *harmonic sheaf*) of continuous functions for open sets D in $\mathbb{R}^d$. Then $(\mathbb{R}^d, {}_{\mu}H)$ forms a *Brelot space* (cf. [6], [10], etc.) if (cf. [3], [4]) μ is of *Kato class* on $\mathbb{R}^d$ characterized by

(2.6)
$$\lim_{\varepsilon \to 0} \kappa(B(a,\varepsilon);\mu) = 0$$

for every $a \in \mathbb{R}^d$ with

(2.7)
$$\kappa(B(a,\varepsilon)) = \kappa(B(a,\varepsilon);\mu) = \sup_{x \in \mathbb{R}^d} \int_{B(a,\varepsilon)} g(x-y)d\,|\mu|(y),$$

where $B(a,\varepsilon)$ is the open ball with radius $\varepsilon > 0$ centered at a and $|\mu|$ is the total variation measure associated with μ. Especailly, for Radon measures μ of constant sign (i.e. for $\mu = |\mu|$ or for $\mu = -|\mu|$), $(\mathbb{R}^d, {}_{\mu}H)$ forms a Brelot space if and only if μ is of Kato class (cf. e.g. [11]). The constant $\kappa(B(a,\varepsilon))$ in (2.7) is referred to as the *Kato constant* of $B(a,\varepsilon)$ (for μ). It is readily seen that μ is of Kato class if and only if the potential $U^{|\mu||K} \in C(\mathbb{R}^d)$ for every compact subset K of $\mathbb{R}^d$. The classical harmonic functions (i.e. continuous solutions of $-\Delta u = (-\Delta + 0)u = 0$) are simply 0-harmonic functions. We simply write $H(D)$ to mean ${}_0H(D)$.

In this paper we only need the case of *positive* Radon measure μ of Kato class. In addition to the class ${}_{\mu}H(D)$ we consider the class ${}_{\mu}S(D)$ of μ-superharmonic (i.e. superharmonic with respect to the harmonic structure ${}_{\mu}H$) functions on D. We also simply write $S(D)$ for ${}_0S(D)$. For a bounded domain V in $\mathbb{R}^d$ we denote by ${}_{\mu}H_f^V$ the generalized solution on V of the Dirichlet problem for the boundary function $f \in C(\partial V)$ with respect to the μ-harmonic structure ${}_{\mu}H$ in the sense of Perron-Wiener-Brelot. A boundary point $y \in \partial V$ is said to be *μ-regular* if $\lim_{x \in V, x \to y} {}_{\mu}H_f^V(x) = f(y)$ for every $f \in C(\partial V)$. A bounded domain V is said to be μ-regular if every boundary point $y \in \partial V$ is μ-regular. We write H_f^V for ${}_0H_f^V$ and say regular for 0-regular.

We denote by ρV the set of regular points in ∂V with respect to V. In other words, ρV is the set of points $y \in \partial V$ such that the complement V^c of V is thick (i.e. not thin) at y. By the Bouligand theorem we have

$$\rho V = \left\{ y \in \partial V : \lim_{x \in V, x \to y} G^V(x,z) = 0 \right\}$$

for any fixed $z \in V$, where $G^V(x,z)$ is the Green function on V. Moreover, $G^V(\cdot,z)$ is extended to be in $C(V \cup \rho V)$ and the extended $G^V(\cdot,z)|\rho V = 0$ for any fixed $z \in V$. We now moreover assume that V is a bounded Carathéodory domain. If we denote by $(G^V)^*(x,y)$ the extended Green function on V, then, since

$$\sigma V = \left\{ x \in \partial V : (G^V)^*(x,y) = 0 \right\}$$

with $(G^V)^*(\cdot,y) \in C(V \cup \sigma V)$ for any fixed point $y \in V$ and $(G^V)(\cdot,y) \leq (G^V)^*(\cdot,y)$ on V, we see that

$$\sigma V \subset \rho V.$$

For any bounded domain V in R^d with its Green function $G^V(x,y)$, we associate a function $T_V \varphi$ on V with a bounded Borel function φ on V given by

(2.8) $$T_V \varphi(x) := \int_V G^V(x,y)\varphi(y) d\mu(y).$$

By the boundedness of φ on V and by the fact that φ is of Kato class, we see that $T_V \varphi$ is bounded on V and $T_V \varphi \in C(V)$. Moreover $T_V \varphi$ can be continued to a continuous function on $V \cup \rho V$ and vanishes on $\rho V : T_V \varphi \in C(V \cup \rho V)$, $T_V \varphi \mid \rho V = 0$. To prove this take an arbitrary positive number ε. By (2.6) we can find, for any fixed $z \in \rho V$, a ball $B := B(z,\delta)$ $(\delta > 0)$ such that

$$| T_V \varphi(x) | \leq \varepsilon + \int_{V \setminus B} G^V(x,y) | \varphi(y) | d\mu(y) \qquad (x \in B \cap V).$$

Since $G^V(x,y) \leq g(x-y) + \text{const.}$ on $V \times V$ and $G^V(x,y) \to 0$ $(x \to z)$ for any fixed $y \in V \setminus B$, letting $x \to z$ in the above inequality, we deduce $\limsup_{x \to z} | T_V \varphi(x) | \leq \varepsilon$ so that $\lim_{x \to z} T_V \varphi(x) = 0$.

If $\varphi \geq 0$ on V, then $T_V \varphi \in S(V)^+ = \{s \in S(V) : s \geq 0 \text{ on } V\}$. By the definition of the Green function, we have

(2.9) $$\Delta T_V u = -\varphi\mu$$

in the sense of distributions. As an important consequence of this we have the following property :

(2.10) $$u + T_V u \in H(V) \qquad (u \in {}_\mu H(V)).$$

In fact, since $\Delta u = u\mu$ and $\Delta T_V u = -u\mu$ by (2.9) on V in the distributional sense. Hence $\Delta(u + T_V u) = 0$ in the sense of distributions. The Weyl lemma and the continuity of $u + T_V u$ assures that $u + T_V u \in H(V)$.

As a result of $\mu \geq 0$ we hav$S(D)^+ \subset {}_\mu S(D)^+$ but we only need the follwoing restricted version of this in the present paper :

(2.11) $$S(D)^+ \cap C(D) \subset {}_\mu S(D)^+ \cap C(D) \qquad (\mu \geq 0)$$

for any open set D in R^d. To see this let s be an arbitrary element in $S(D)^+ \cap C(D)$. We need to show that $s \geq {}_\mu H_S^V$ for every μ-regular domain V with $\overline{V} \subset D$. Set $u := {}_\mu H_S^V$. By (2.10) $u + T_V u$ is a bounded harmonic function on V taking the boundary values s on ρV. From this it follows that $H_S^V = u + T_V u$ and a fortiori $H_S^V \geq {}_\mu H_S^V$ on V since $T_V u \geq 0$ on V. By $s \in S(D)^+$, we have $s \geq H_S^V$ on V and thus $s \geq {}_\mu H_S^V$ on V.

2.4. Proof of Theorem. We denote by $\text{cap}(E)$ the (outer) capacity of a set $E \subset \mathrm{R}^d$

so that $\text{cap}(E)$ is the logarithmic capacity of E for $d=2$ and the Newtonean capacity of E for $d\geq 3$. Suppose first that $\text{cap}(\partial M\backslash\sigma M)=0$. Then the harmonic measure of $\partial M\backslash\sigma M$ relative to M is zero and hence the Keldysh theorem asuures the stability of the Dirichlet problem in M. Thus only the proof of the necessity of the condition $\text{cap}(\partial M\backslash\sigma M)=0$ is nontrivial. Hence we suppose that the Dirichlet problem is stable in M and yet $\text{cap}(\partial M\backslash\sigma M)=0$. From this erroneous assumption $\text{cap}(\partial M\backslash\sigma M)>0$ we will derive a contradiction.

We choose and then fix a squeezer $(M_n)_{n\geq 1}$ of $\overline{M}$ such that M_n are bounded regular regions in R^d and $M_n\subset M_{n+1}(n\geq 1)$. We denote by $G_n(x,y)$ ($G(x,y)$, resp.) the Green function on M_n (M, resp.) for every $n\geq 1$. Then $G_n(x,y)\downarrow G^*(x,y)\geq G(x,y)$ as $n\uparrow\infty$, where $G^*(x,y)$ is the external Green function on M. Fix an arbitrary $y\in M$. Since $G_n(\cdot,y)\in C(\partial M)$ and $G_n(\cdot,y)\downarrow G^*(\cdot,y)$ on ∂M, $G^*(\cdot,y)$ is upper semicontinuous. By (2.4), $\partial M\backslash\sigma M=\{x\in\partial M: G^*(x,y)>0\}$ so that $\partial M\backslash\sigma M$ is a Borel set. By the Choquet capacitability theorem (cf. e.g. [7, p.149]) there exists a compact set $F\subset\partial M\backslash\sigma M$ such that $\text{cap}(F)>0$. Denote by $\mathcal{M}_F^+$ the class of positive Radon measures ν with $\nu\subset F$, where spt ν is the support of ν. Then since

$$\text{cap}(F)=\sup\{\nu(F):\nu\in\mathcal{M}_F^+, U^\nu\leq 1 \text{ on } \mathrm{R}^d\}$$

(cf.e.g. [7, p.140]), we can find and then fix a $\nu\in\mathcal{M}_F^+$ such that $\nu(F)>0$ and $U^\nu\leq 1$ on R^d. By an application of the Lusin theorem we can find a compact set $K\subset F$ such that $\nu(F\backslash K)<\nu(F)/2$ and $U^{\nu|K}\in C(\mathrm{R}^d)$, i.e. $\nu|K$ is of Kato class (cf. e.g. [7, p.118]). We have thus established the existence of a positive Radon measure μ (e.g. $\mu=\nu|K$) of Kato class on R^d satisfying

$$\mu(\partial M\backslash\sigma M)>0,\qquad \text{spt } \mu\subset\partial M\backslash\sigma M. \tag{2.12}$$

In the sequel we will derive a contradiction that $\mu(\partial M\backslash\sigma M)=0$.

We first note that each M_n is μ-regular as a result of the assumption that M_n is regular (in reality, we know as a general result that the regularity of M_n and the μ-regularity of M_n for any (signed) Radon measure μ of Kato class are equivalent, the full statement of which will not be needed in our proof). This can be seen, for example, as follows. Using the Green function $G_n(x,y)$ on M_n we set $s:=\min(G_n(\cdot,y),\ 1)>0$ for a fixed $y\in M_n$, which belongs to $S(M_n)^+\cap C(\overline{M}_n)$ and $s|\partial M_n=0$. Thus, by (2.11), we see that $s\in{}_\mu S(M_n)^+\cap C(\overline{M}_n)$ and $s>0$ with $s|\partial M_n=0$ so that s is a μ-barrier for each boundary point of ∂M_n. Hence the existence of the μ-barrier assures the μ-regularity of M_n (cf. e.g. [6, p.57]).

For each $n\geq 1$ we consider $u_n:={}_\mu H_1^{M_n}$, which, by the μ-regularity of M_n, belongs to ${}_\mu H(M_n)\cap C(\overline{M}_n)$ with $u_n|\partial M_n=1$. In view of (2.11) the constant function $1\in{}_\mu S(\mathrm{R}^d)$.

Hence by the minimum principle we see that $0 < u_1 \le u_n \le u_{n+1} \le 1$ on M_{n+1} and therefore $(u_n)_{n\ge 1}$ forms an increasing sequence dominated by 1 on $\overline{M}$. We can thus define

$$u^*(x) := \lim_{n\to\infty} u_n(x) \qquad (x \in \overline{M}),$$

which satisfies $0 < u_1 \le u^* \le 1$ on $\overline{M}$. Since $u_n \mid M \in {}_{\mu}H(M)$ for each $n \ge 1$, we see that $u^* \mid M \in {}_{\mu}H(M)$. Observe that, by (2.10), $u_n + T_{M_n} u_n \in H(M_n)$ and by examining the boundary behavior of $T_{M_n} u_n$ at ∂M_n we see that $u_n + T_{M_n} u_n \in C(\overline{M}_n)$ with $u_n + T_{M_n} u_n = 1$ on ∂M_n. Hence $u_n + T_{M_n} u_n \equiv 1$ on M_n and, in particular,

(2.13) $$1 = u_n(x) + \int_{\mathrm{spt}\mu} G_n(x,y) u_n(y) d\mu(y) \qquad (x \in M).$$

Note that $G_n(x,y) \downarrow G^*(x,y)\,((x,y) \in M \times \partial M)$, $0 < G_n(x,y)u_n(y) \le G_1(x,y) \le g(x-y) + \text{const.}$ $((x,y) \in M \times \partial M)$, and $g(x-y)$ is μ-integrable as the function of y on ∂M for any fixed $x \in M$. Therefore, by the Lebesgue dominated convergence theorem, on letting $n \uparrow \infty$ in (2.13) we obtain

(2.14) $$1 = u^*(x) + \int_{\mathrm{spt}\mu} G^*(x,y) u^*(y) d\mu(y) \qquad (x \in M).$$

Since $M \cap (\mathrm{spt}\ \mu) = \emptyset$, $u^* \mid M \in {}_{\mu}H(M) = H(M)$ so that $u^* \mid M$ is a bounded harmonic function on M. Fix an arbitrary $a \in \sigma M$. By the remark to (2.3) we see that $(x,y) \longmapsto G^*(x,y)$ is continuous at (a,b) $(b \in \mathrm{spt}\ \mu)$ as the function on $\overline{M} \times \mathrm{spt}\ \mu$ and $G^*(a,b) = 0$ since $\mathrm{spt}\ \mu \subset \partial M \backslash \sigma M$. Since we are assuming that the Dirichlet problem is stable in M, the Keldysh theorem assures that the harmonic measure of $\partial M \backslash \sigma M$ relative to M is zero. We have thus seen by (2.14) that the bounded harmonic function $u^* \mid M$ on M has boundary values 1 on ∂M except for a set of points in the set $\partial M \backslash \sigma M$ of harmonic measure zero relative to M. This implies that $u^* \mid M \equiv 1$ on M. Hence by (2.14) we deduce

(2.15) $$\int_{\partial M \backslash \sigma M} G^*(x,y) u^*(y) d\mu(y) = 0 \qquad (x \in M).$$

Here $G^*(x,\cdot) > 0$ on $\partial M \backslash \sigma M$ for any fixed $x \in M$ (cf. (4)); $u^* \ge u_1 > 0$ on M and of course on $\partial M \backslash \sigma M$. Therefore (2.15) implies $\mu(\partial M \backslash \sigma M) = 0$, which contradicts the starting assumption that $\mu(\partial M \backslash \sigma M) > 0$ (cf. (2.12)).

3. The proof based on the theory of balayage

In this Section 3 we will prove our main theorem by making essential use of the operation called balayage or sweeping out. As in Section 2 the external Green function again also plays an important role in the proof described in this section. Thus in subsection 3.1. *External Green functions* with the same title as in subsection 2.1 we give the definition of external Green functions and their important properties used in the proof from the view point of balayage operations. In the final subsection 3.1. *Proof of Theorem* we give relatively short proof of our main theorem by positively using the balayage method.

3.1. External Green functions. Since we have assumed that the region M is bounded in R^d, we can find and then fix a finite open ball

$$\Omega$$

containing $\overline{M}$. For any Borel set $A \subset \Omega$ and nonnegative superharmonic function u on Ω, we denote by $\hat{R}_u^A$ the balayage of u on A considered for the space Ω. We also denote by μ^A the balayage of a Borel measure μ on A considered for Ω. For a subregion $D \subset \mathrm{R}^d$ we denote by $G_D(x,y)$ the Green function on D and in particular we set

$$G(x,y) := G_\Omega(x,y)$$

for simplicity. For a Borel measure μ on Ω we denote by G^μ the Green potential on Ω of the measure $\mu: G^\mu(x) = \int_\Omega (x,y)d\mu(y)$.

For any squeezer $(M_i)_{i \geq 1}$ of $\overline{M}$, $(G_{M_i}(x,y))_{i \geq 1}$ is seen to converge to a function, denoted by $G_{\overline{M}}(x,y)$, for x and y in $\overline{M}$, which is independent of the choice of the squeezer $(M_i)_{i \geq 1}$:

(3.1) $$G_{\overline{M}}(x,y) = \lim_{i \uparrow \infty} G_{M_i}(x,y) \qquad ((x,y) \in \overline{M} \times \overline{M})$$

which is referred to as the *external Green function* of M. As the limit of symmetric functions $G_M(x,y)$, $G_{\overline{M}}(x,y)$ is symmetric : $G_{\overline{M}}(x,y) = G_{\overline{M}}(y,x)$. The set σM of stable points in ∂M can be characterized in terms of the external Green function as follows (cf. [9, p.333]) :

(3.2) $$\sigma M = \{y \in \partial M : G_{\overline{M}}(x,y) = 0\}$$

for one and hence for every fixed $x \in M$. A squeezer $(M_i)_{i \geq}$ of $\overline{M}$ will be said to be special if each M_i is regular and $\overline{M} \subset M_{i+1} \subset \overline{M}_{i+1} \subset M_i \subset \overline{M}_i \subset \Omega \, (1 \leq i < \infty)$. Taking a special squeezer $(M_i)_{i \geq 1}$ of $\overline{M}$ in (1), we see that $G_{\overline{M}}$ is upper semicontinuous on $\overline{M} \times \overline{M}$ as the limit of decreasing sequence of continuous functions G_{M_i} on $\overline{M} \times \overline{M}$. Hence we can conclude that σM in (3.2) is a Borel set as announced earlier.

Observe on the other hand that $G_{M_i}(\cdot,y) = G(\cdot,y) - H_{G(\cdot,y)}^{M_i}$ and $H_{G(\cdot,y)}^{M_i} = \hat{R}_{G(\cdot,y)}^{\Omega \setminus M_i}$ on $\overline{M}$ for any $y \in \overline{M}$. Since $(\Omega \setminus M_i)_{i \geq 1}$ is increasing and converges to $\cup_{i \geq 1}(\Omega \setminus M_i) = \Omega \setminus \cap_{i \geq 1} M_i = \Omega \setminus \overline{M}$, by a property of balayage of functions (cf. e.g. [2, p.246], [6, p.114]), $(\hat{R}_{G(\cdot,y)}^{\Omega \setminus M_i})_{i \geq 1}$ converges to $\hat{R}_{G(\cdot,y)}^{\Omega \setminus \overline{M}}$ on Ω. Hence, in particular, we obtain another representation of the external Green function of M :

(3.3) $$G_{\overline{M}}(x,y) = G(x,y) - \hat{R}_{G(\cdot,y)}^{\Omega \setminus \overline{M}}(x) \qquad ((x,y) \in \overline{M} \times \overline{M}).$$

3.2. Proof of Theorem. We denote by $\mathrm{cap}(E)$ the (outer) capacity of a set $E \subset \mathrm{R}^d$ so that $\mathrm{cap}(E)$ is the logarithmic capacity of E for $d = 2$ and the Newtonean capacity of

E for $d \geq 3$. Suppose first that $\operatorname{cap}(\partial M \backslash \sigma M)=0$. Then the haromonic measure of $\partial M \backslash \sigma M$ relative to M is zero and hence the Keldysh theorem assures the stability of the Dirichlet problem in M. Thus only the proof of the necessity of the condition $\operatorname{cap}(\partial M \backslash \sigma M)=0$ is nontrivial. We originally proved this part as an application of the theory of Schrödinger equations with potentials of Kato measures; for the following direct simple proof given below, we owe a lot to Professor Arnaud de la Pradelle to whom we are deeply grateful. The proof for the necessity given below is by contradiction. Namely, we suppose that the Dirichlet problem is stable in M and yet $\operatorname{cap}(\partial M \backslash \sigma M)>0$. From this erroneous assumption $\operatorname{cap}(\partial M \backslash \sigma M)>0$ we will derive a contradiction.

Recall that $\partial M \backslash \sigma M$ is a Borel set. By the Choquet capacitability theorem (cf. e.g. [7, p.149]) there exists a compact set $F \subset \partial M \backslash \sigma M$ such that $\operatorname{cap}(F)>0$ since $\operatorname{cap}(\partial M \backslash \sigma M)>0$. Denote by $\mathcal{M}_F^+$ the class of positive Radon measures ν with $\operatorname{spt} \nu \subset F$, where $\operatorname{spt} \nu$ is the support of ν. Then since

$$\operatorname{cap}(F)=\sup\{\nu(F): \nu \in \mathcal{M}_F^+,\ G^\nu \leq 1 \text{ on } \Omega\}$$

(cf. e.g. [7, p.140]), we can find and then fix a $\nu \in \mathcal{M}_F^+$ such that $\nu(F)>0$ and $G^\nu \leq 1$ on Ω. By an application of the Lusin theorem we can find a compact set $K \subset F$ such that $\nu(F \backslash K)<\nu(F)/2$ and $G^{\nu|K} \in C(\Omega)$ (cf. e.g. [7, p.118]). We have thus established the existence of a positive Radon measure μ (e.g. $\mu=\nu \mid K$) on Ω satisfying

(3.4) $$\mu(\partial M \backslash \sigma M)>0, \qquad \operatorname{spt} \mu \subset \partial M \backslash \sigma M, \qquad G^\mu \in C(\Omega).$$

In the sequel we will derive a contradiction that $\mu(\partial M \backslash \sigma M)=0$.

Since G^μ is contiuous on Ω and harmonic on M, we obtain $G^\mu=H_{G^\mu}^M$ on M. By the assumption that the Dirichlet problem is stable inside M, we have $H_f^M=H_f^{\overline{M}}$ on M for any $f \in C(\partial M)$, and in particular, by taking $f=G^\mu \in C(\partial M)$, we see that $H_f^M=H_f^{\overline{M}}$ on M. The latter is, by definition, the limit of the sequence $(H_G^{M_i})_{i \geq 1}$ for a special squeezer $(M_i)_{i \geq 1}$ of $\overline{M}$. Since M_i is regular, $H_{G^\mu}^{M_i}=R_{G^\mu}^{\Omega \backslash M_i}$ on M_i, and hence on M. By a property of balayage of functions already used in subsection 3.2, since $(\Omega \backslash M_i)_{i \geq 1}$ is increasing and $\cup_{i \geq 1}(\Omega \backslash M_i)=\Omega \backslash \overline{M}$, we see that $\lim_{i \uparrow \infty} \hat{R}_{G^\mu}^{\Omega \backslash M_i}=\hat{R}_{G^\mu}^{\Omega \backslash \overline{M}}$ on Ω so that we have deduced the follwing string of identities:

(3.5) $$G^\mu=H_{G^\mu}^M=H_{G^\mu}^{\overline{M}}=\lim_{i \uparrow \infty} H_{G^\mu}^{M_i}=\lim_{i \uparrow \infty} \hat{R}_{G^\mu}^{\Omega \backslash M_i}=\hat{R}_{G^\mu}^{\Omega \backslash \overline{M}}$$

on M. As a nonnegative superharmonic function dominated by G^μ on Ω, $\hat{R}_{G^\mu}^{\Omega \backslash \overline{M}}$ is a Green potential G^λ of a measure λ concentrated on $\overline{\Omega \backslash \overline{M}}$. Observe that G^λ is the minimal Green potential satisfying $G^\lambda \leq G^\mu$ on Ω, $G^\lambda=G^\mu$ quasieverwhere on $\Omega \backslash \overline{M}$, and λ is concentrated on $\overline{\Omega \backslash \overline{M}}$. This characterizes the balayaged measure $\mu^{\Omega \backslash \overline{M}}$ (cf. e.g. [5, p.50], [7, p.237], [9, p.274], etc.) so that $\lambda=\mu^{\Omega \backslash \overline{M}}$ or

(3.6) $$\hat{R}_{G^\mu}^{\Omega \backslash \overline{M}}(x)=G^{\mu^{\Omega \backslash \overline{M}}}(x) \qquad (x \in \Omega).$$

By a relation between balayage of functions and measures (cf. e.g. [2, p.255], [6, p.160]), we have

$$\int_\Omega G(x,y)\,d\mu^{\Omega\setminus\overline{M}}(y) = \int_\Omega \hat{R}^{\Omega\setminus\overline{M}}_{G(x,\cdot)}(y)\,d\mu(y) = \int_\Omega \hat{R}^{\Omega\setminus\overline{M}}_{G(\cdot,x)}(y)\,d\mu(y)$$

for ever $x\in\Omega$. By (3), the symmetry of $G_{\overline{M}}(x,y)$ and $G(x,y)$ for $(x,y)\in\overline{M}\times\overline{M}$ implies that of $\hat{R}^{\Omega\setminus\overline{M}}_{G(\cdot,x)}(y)$, i.e. $\hat{R}^{\Omega\setminus\overline{M}}_{G(\cdot,x)}(y)=\hat{R}^{\Omega\setminus\overline{M}}_{G(\cdot,y)}(x)$, and therefore

(3.7) $$G^{\mu^{\Omega\setminus\overline{M}}}(x) = \int_\Omega G(x,y)\,d\mu^{\Omega\setminus\overline{M}}(y) = \int_{\mathrm{spt}\,\mu} \hat{R}^{\Omega\setminus\overline{M}}_{G(\cdot,y)}(x)\,d\mu(y)$$

since spt $\mu\subset\overline{M}$. By (3.5), (3.6) and (3.7) we conclude that

$$\int_\Omega G(x,y)\,d\mu(y) = \int_\Omega \hat{R}^{\Omega\setminus\overline{M}}_{G(\cdot,y)}(x)\,d\mu(y)$$

for every $x\in M$. This with (3.3) implies that

$$\int_{\partial M\setminus\sigma M} G_{\overline{M}}(x,y)\,d\mu(y) = 0 \qquad (x\in\overline{M}).$$

By (3.2), the integrand $G_{\overline{M}}(x,\cdot)>0$ on $\partial M\setminus\sigma M$ for any fixed $x\in M$ so that we must have $\mu(\partial M\setminus\sigma M)=0$, contradicting (3.4).

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4. APPENDIX : An abstract for one hour talk

The follwoing is the exact copy of the abstract distributed to the attendance of our lecture delibered at a domestic meeting held in Osaka. The language used there was Japanese and therefore the abstract itself is also written in Japanese. The mathematical content of the abstract below is almost identical with that in§2 of the present paper but the explanation is more unaffected and naive and also there is an additional information not in§2. Therefore it may be useful to record it here, which is the reason of the inclusion of this section in this paper.

ポテンシャル論研究集会
1999年11月25日－27日
大阪市立大学学術情報センター
10階会議室

ディリクレ問題の安定領域

１．序論

d 次元ユークリッド空間 $\mathrm{R}^d\,(d\geq 2)$ 内の有界カラテオドリー領域Mを考える，即ち $\partial\overline{M}=\partial M$となるものとする．有界な領域$M_n$の列$(M_n)_{n\geq 1}$が

$$M_n\supset\overline{M},\qquad \partial M_n\to\partial M$$

を満たすとき$\overline{M}$の**圧搾列**と言う，但し上の後者はどんな$\delta>0$に対しても或番号$n(\delta)$が定まってすべての番号$n\geq n(\delta)$にたいして∂M_nと∂Mのいずれも他方のδ‐近傍に含まれることを意味する．どんな$f\in C(\partial M)$も$f\in C(\mathrm{R}^d)$となる様に拡張できる．どんな$f\in C(\mathrm{R}^d)$も$\overline{M}$の近傍で２つの連続優調和関数の差で一様近似されることを使えば**外ディリクレ解**

$$H_f^{\overline{M}}(x)=\lim_{n\to\infty}H_f^{M_n}(x)\qquad (x\in\overline{M})$$

が存在して，圧搾列$(M_n)_{n\geq 1}$の取り方にも$f\in C(\partial M)$のR^dへのどんな連続拡張にも依存せず$\overline{M}$と$f\in C(\partial M)$のみにより一意に定まることが容易にわかる．どんな$f\in C(\partial M)$に対してもM上$H_f^{\overline{M}}=H_f^M$となるとき，ディリクレ問題はM内で**安定**であると言う．どんな$f\in C(\partial M)$に対しても$H_f^{\overline{M}}(y)=f(y)$となる様な点$y\in\partial M$は$M$の**安定点**であると言い，その全体を記号

$$\sigma M$$

で表す．勿論$y\in\partial M\backslash\sigma M$は非安定点であると言う．$y\in\partial M$が$y\in\sigma M$となる為の必要十分な条件は$\overline{M}^c=R^d\backslash\overline{M}$が$y$に於いて肥厚(即ち非尖細)なことである([6], [7, p. 308])．Mの正

則点の全体を ρM と記すならば，$y \in \partial M$ が $y \in \rho M$ となる為の必要十分条件は $M^c = R^d \backslash M$ が y に於いて肥厚となることであるので

$$\sigma M \subset \rho M$$

となる．次の有名な古い結果を想起する（[6]，[7, p.340]）：

ケルディシュの定理．ディリクレ問題が M 内安定である為の必要重十分条件は $\partial M \backslash \sigma M$ が M に関する調和測度零となることである．

本講演の目的は上記定理に於ける $\partial M \backslash \sigma M$ が調和測度零となると言う条件を容量零となると言う条件で置き換え得ることを示すことである．ここに容量は $d=2$ ならば対数容量 $d \geq 3$ ならばニュートン容量を意味するものとする．即ち次の結果が成り立つ（[10]）：

定理．ディリクレ問題が M 内安定である為の必要十分条件は $\partial M \backslash \sigma M$ が容量零となることである．

2．外ディリクレ関数

M_n のグリーン関数を $G_n(x,y)$ とするとき，領域拡大の原理により

(1) $$G^*(x,y) := \lim_{n \to \infty} G_n(x,y) \qquad ((x,y) \in \overline{M} \times \overline{M})$$

の存在がわかる．これを M の**外グリーン関数**と呼ぶ．G_n の諸性質のいくつかは自然に G^* に遺伝する．例えば，$G^*(\cdot,y)(y \in M)$ は $M \backslash \{y\}$ 上正値調和関数で y で基本調和極を持つとか，$G^*(x,y) = G^*(y,x)((x,y) \in \overline{M} \times \overline{M})$ とかである．勿論 M のグリーン関数 G とは

$$G^*(x,y) \geq G(x,y) \qquad ((x,y) \in M \times M)$$

の関係にあり，ディリクレ問題が M 内安定なら上で等号が成立するが $G(\cdot,y)(y \in M)$ の ∂M に於ける境界挙動（例えば ρM で境界値零）と $G^*(\cdot,y)$ の ∂M に於ける値は一般に一致しない（例えば ρM 上 $G^*(\cdot,y) = 0$ とは限らない）．R^d の基本調和関数 $g(x)$ を使って R^d 上のラドン測度 μ の基本ポテンシャルを

$$U^\mu(x) := g * \mu(x) = \int_{\mathrm{spt}\,\mu} g(x-y)\,d\mu(y)$$

と書くことにすると（但し $\mathrm{spt}\,\mu$ は μ の台）

(2) $$G^*(x,y) = U^{\varepsilon_y}(x) - U^{\beta_{\overline{M}^c}\varepsilon_y}(x)$$

と表せる．ここで，ε_y は台が $y \in \overline{M}$ にあるディラック測度で，$\beta_{\overline{M}^c}$ は $\overline{M}$ 上の測度から $\overline{M}^c$ への掃散作用素とする．此の表示を使うと M のグリーン関数 G に対するケロッグの定理に相当する次の重要な G^* の境界挙動が示される（[7, p.333] 参照）：任意の $b \in M$ を固定するとき，$a \in \partial M$ に対して，$a \in \sigma M$ となる必要十分条件は

(3) $$G^*(a,b) = 0$$

である；更に $b \in \overline{M}$ を固定するとき，$a \in \sigma M \backslash \{b\}$ なら(3)が成り立ち，そのとき関数

$(x,y) \longmapsto G^*(x,y)$は$\overline{M}\times\overline{M}$ の関数として，(a,b) に於いて連続となる(上記参照書物では $b\in M$ の時だけが示されているが簡単なトリックで $b\in\partial M$ の場合も含めることが出来る).

3．シュレーディンガー方程式のブルロー空間

R^d 上の一般符号ラドン測度 μ をポテンシャル項に持つ定常シュレーディンガー方程式

(4) $$(-\Delta+\mu)u=0$$

の関集合 D 上の連続超関数解 u を D 上のμ-**調和関数**と言いその全体を${}_\mu H(D)$と記す. ${}_\mu H: D \to {}_\mu H(D)$は$\mathrm{R}^d$上の$\mu$-**調和層**と呼ばれる連続関数の層を作る．そのとき$(\mathrm{R}^d,\ {}_\mu H)$が**ブルロー空間**([4]，[8] 等参照)となる十分条件は μ が**カトー族**の測度であることである([2]，[3] 参照)，ここで μ がカトー族とはすべての $a\in\mathrm{R}^d$ に対し

(5) $$\lim_{\varepsilon\downarrow 0}\left(\sup_{x\in\mathrm{R}^d}\int_{B(a,\varepsilon)} g(x-y)d\,|\,\mu\,|(y)\right)=0$$

となることとする，但し$B(a,\varepsilon)$は a 中心半径 $\varepsilon>0$ のR^d内の開球とする．此の条件はR^dの全てのコムパクト集合Kに対して$U^{|\mu\|K}\in C(\mathrm{R}^d)$となる事と同値である．特に μ を定符号(正値又は負値)とするとき，$(\mathrm{R}^d,\ {}_\mu H)$がブルロー空間となる為の必要十分条件は μ がカトー族となることである([9]).

さてμ をカトー族の正値ラドン測度とする．正値性の帰結として$1\in{}_\mu S(\mathrm{R}^d)^+$(正値$\mu$-優調和関数(即ち${}_\mu H$に関する優調和関数)の全体)となる．$f\in C(\partial D)$に対して${}_\mu H$に関するペロン・ウィーナー・ブルローの方法によるディリクレ問題の解を${}_\mu H_f^D$と記す．$y\in\partial D$ のμ-正則性と正則性(即ち 0- 正則性)は一致することが示される.

4．証明の概要

条件の必要性を示すために$\mathrm{cap}(\partial M\setminus\sigma M)>0$として矛盾を出す．すると$\mathrm{R}^d$上のカトー族の正値ラドン測度$\mu$ で$\mu(\partial M\setminus\sigma M)>0$かつ$\mathrm{spt}\ \mu\subset\partial M\setminus\sigma M$となるものがとれる．$\overline{M}$の圧搾列$(M_n)_{n\geq 1}$で，$M_n\supset M_{n+1}$かつ$M_n$が${}_0H$に関しても${}_\mu H$に関しても正則なものがとれる．1が${}_\mu H$に関して優調和なことから$u_n := {}_\mu H_1^{M_n}$の列が単調増加で$u^* := \lim_{n\to\infty} u_n$が$\overline{M}$上存在し，$\overline{M}$上$0\leq u^*\leq 1$かつ$u^*\,|\,M\in{}_\mu H(M)$となる． ${}_0H_1^{M_n}\equiv 1$と${}_\mu H_1^{M_n}$の間の摂動等式(レゾルベント方式)により

$$1=u_n(x)+\int_{\mathrm{spt}\,\mu} G_n(x,y)u_n(y)d\,\mu(y) \qquad (x\in M)$$

となり，その極限として

$$1=u^*(x)+\int_{\mathrm{spt}\,\mu} G^*(x,y)u^*(y)d\,\mu(y) \qquad (x\in M)$$

が得られる．ディリクレ問題がM内安定であると，$\partial M\setminus\sigma M$の調和測度が零となるから上式より，$u^*\equiv 1$となることがわかり

$$\int_{\partial M\setminus\sigma M} G^*(x,y)u^*(y)d\,\mu(y)=0 \qquad (x\in M)$$

となる．$\partial M\setminus\sigma M$上$G^*(x,\cdot)>0,\ u^*\geq u_1>0$だから，上式より$\mu(\partial M\setminus\sigma M)=0$と言う矛盾が出

る．

5．用語についての注意

ディリクレ問題がカラテオドリー領域Mで安定（又別に$\overline{M}$で安定，即ち全ての$f \in C(\partial M)$に対して$H_f^{M_n}$が$H_f^{\overline{M}}$に$\overline{M}$上一様収束すること）と言う用語はケルディシュ[6]以来のものでランドコフ[7]の本でもこれを踏襲している．しかしながら近年調和近似論で，コンパクト集合Kが**安定**と言う用語が，$H(K) = C_h(K)$の意味で，例えばヘドベルグ（L. I. Hedberg : *Approximation by harmonic functions, and stability of the Dirichlet problem*, Expo. Math., **11** (1993), 193－259；又[1]も参照）等が使っている，但し$H(K) = C_h(K)$はKの近傍での調和な関数族の$C(K)$内での一様閉被，$C_h(K)$はKの内部で調和な$C(K)$の関数の全体とする．従って，例えば，Mが正則領域ならば，ディリクレ問題が$\overline{M}$で安定なことと，コムパクト集合$\overline{M}$が安定となることが同値となる．ケルディシュの場合でも，もっと精密に，全ての連続境界値（又は指定された境界値達）に対してディリクレ問題がM（又は$\overline{M}$）で安定と言う言い方もしている．この言い方に従えば，M上ディリクレ問題が真の解を持つ様な全ての連続境界値に対してディリクレ問題が$\overline{M}$上安定なこととコムパクト集合$\overline{M}$が安定なこととが同値となる．

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